| Nº 228 | **COPULA-BASED MEASURES OF DEPENDENCE STRUCTURE IN ASSETS RETURNS** |
|---|---|
| | **Viviana Fernández** |

# DOCUMENTOS DE TRABAJO

## Serie Economía

# Copula-based measures of dependence structure in assets returns

Viviana Fernandez[1]

## Abstract

Copula modeling has become an increasingly popular tool in finance to model assets returns dependency. In essence, copulas enable us to extract the dependence structure from the joint distribution function of a set of random variables and, at the same time, to separate the dependence structure from the univariate marginal behavior. In this study, based on U.S. stock data, we illustrate how tail-dependency tests may be misleading as a tool to select a copula that closely mimics the dependency structure of the data. This problem becomes more severe when the data is scaled by conditional volatility and/or filtered out for serial correlation. The discussion is complemented, under more general settings, with Monte Carlo simulations.

Keywords: copulas, extreme-value dependency.

[1] Associate Professor, Center for Applied Economics (CEA) at the Department of Industrial Engineering of the University of Chile. Postal: Avenida Republica 701, Santiago-Chile. Email: vfernand@dii.uchile.cl; fax: (562) 689-7895. Financial support from FONDECYT Grant No. 1050486 and from an institutional grant of the Hewlett Foundation to CEA is greatly acknowledged. All remaining errors are the author's.

## 1    Introduction

Modeling the dependence structure of assets returns has become an active line of research in finance in research years. In particular, extreme value theory has been applied to model tail dependency. Two recent articles in this area are Poon, Rockinger, and Tawn (2003, 2004). Poon et al. conclude that extreme-value dependence is usually stronger in bearish (left tails) than in bullish markets (right tails), and that some of this dependency can be explained by correlated conditional volatilities.

A more general methodology, which enables us to study not only the tail behavior but the whole structure of dependency of a set of random variables, is copula modeling. Specifically, copulas are uniform distributions, which make it possible to extract the dependence structure from the joint probability distribution function of a set of random variables and, simultaneously, to separate the dependence structure from the univariate marginal behavior. Examples of recent applications of copulas in finance are Cherubini and Luciano (2002, 2003a, b), Embrechts, Lindskog and McNeil (2003), Giesecke (2004), Junker, Szimayer, and Wagner (2005), Pachenko (2005), and Rosenberg and Schuermann (2006). The textbook by Cherubini, Luciano, and Vecchiato (2004) provides a complete discussion on the use of copulas in finance.

In this article, we illustrate how tail-dependency tests may be misleading as a guidance to choose a suitable copula to the data. This can be specially the case when the data is scaled by volatility and/or filtered out for serial correlation. The discussion is illustrated under different scenarios by means of Monte Carlo simulations.

This article is organized as follows. Section 2 presents background material on tail-dependency tests and copulas. Section 3 concentrates on an application of copula

selection to daily data (June 1992-June 2006) of four U.S. stock indices elaborated by Morgan Stanley, namely, U.S. Investable Market Value, U.S. Large Cap 300, U.S. Mid Cap 450, and U.S. Small Cap 1750. By means of Monte Carlo simulations, we look into the issue of how well tail-dependency tests do as a guidance to choose a suitable copula. Section 4 concludes.

## 2 Methodological issues

### 2.1 Tail-dependency tests

### 2.1.1. Asymptotic dependence and asymptotic independence

Focusing exclusively on the probability distribution of the maximum or the minimum of a sample is inefficient if other data on extreme values are available. Therefore, an alternative approach consists of modeling the behavior of extreme values above a high threshold ("Peaks over threshold" or POT). The excess distribution, above a threshold u, is given by the conditional probability distribution

$$F_u(y) = \Pr(X - u \le y \mid X > u) = \frac{F(y+u) - F(u)}{1 - F(u)}, \quad y > 0.$$

(1)

Under some regularity conditions, there exists a positive function $\beta(u)$, for a large enough u, such that (1) is well approximated by the generalized Pareto distribution (GPD):

$$H_{\zeta, \beta(u)}(y) = \begin{cases} 1 - \left(1 + \dfrac{\zeta y}{\beta(u)}\right)^{-1/\zeta} & \zeta \ne 0 \\[2em] 1 - \exp\left(-\dfrac{y}{\beta(u)}\right) & \zeta = 0 \end{cases},$$

(2)

where $\beta(u)>0$, and $y\geq0$ when $\zeta\geq0$, and $0\leq y\leq-\beta(u)/\zeta$ when $\zeta<0$ (see, for example, Coles, 2001, or Embrechts, Klüpperberg and Mikosch, 1997). If $\zeta>0$, F is said to be in the Fréchet family and $H_{\zeta,\beta(u)}$ is a Pareto distribution. In most applications of risk management, the data comes from a heavy-tailed distribution, so that $\zeta>0$.

Poon, Rockinger, and Tawn (2003, 2004) introduce a special case of threshold modeling connected with the generalized Pareto distribution, for the Fréchet family. For this particular case, the tail of a random variable Z above a (high) threshold u can be approximated as

$$1-F(z) = \Pr(Z > z) \sim z^{-1/\eta}L(z), \qquad \text{for } z>u$$

(3)

where $L(z)$ is a slowly varying function of z,[2] and $\eta>0$. If $L(z)$ is treated as a constant for all $z>u$, such that $L(z)=c$, and under the assumption of n independent observations, the maximum-likelihood estimators of $\eta$ and c are

$$\hat{\eta} = \frac{1}{n_u}\sum_{j=1}^{n_u}\log\left(\frac{z_{(j)}}{u}\right) \qquad \hat{c} = \frac{n_u}{n}u^{1/\hat{\eta}},$$

(4)

where $z_{(1)},\ldots, z_{(n_u)}$, are the $n_u$ observations above the threshold u, and $\hat{\eta}$ is known as the Hill estimator.

In order to study dependency of paired returns, Poon et al. suggest transforming the original variables to a common marginal distribution. If (X,Y) are bivariate returns with corresponding cumulative distribution functions $F_X$ and $F_Y$, the following transformation turns them into unit Fréchet marginals (S, T):

---

[2] A function on L on $(0,\infty)$ is slowly varying if $\lim_{z\to\infty} L(tz)/L(z)=1$ for $t>0$.

$$S = -\frac{1}{\ln F_X(X)} \qquad\qquad T = -\frac{1}{\ln F_Y(Y)} \qquad S>0, T>0.$$

(5)

Under this transformation, $\Pr(S>s)=\Pr(T>s)\sim s^{-1}$. As both S and T are on a common scale, the events $\{S>s\}$ and $\{T>s\}$, for large values of s, correspond to equally extreme events for each one. Given that $\Pr(S>s)\to 0$ as $s\to\infty$, the focus of interest is the conditional probability $\Pr(T>s|S>s)$, for large s. If (S, T) are perfectly dependent, $\Pr(T>s|S>s)=1$. By contrast, if (S, T) are exactly independent, $\Pr(T>s|S>s)=\Pr(T>s)$, which tends to zero as $s\to\infty$. Poon et al. define the following measure of asymptotic dependence:

$$\chi = \lim_{s\to\infty} \Pr(T>s \,|\, S>s) \qquad\qquad 0\le\chi\le 1 \;.$$

(6)

In particular, two random variables are called asymptotically dependent if $\chi>0$, and asymptotically independent if $\chi=0$.

Coles, Heffernan and Tawn (1999) point out that two random variables, which are asymptotically independent (i.e., $\chi=0$), may show, however, different degrees of dependence for finite levels of s. Therefore, they propose the following measure of asymptotic independence:

$$\bar{\chi} = \lim_{s\to\infty} \frac{2\log(\Pr(S>s))}{\log(\Pr(S>s,T>s))} - 1 \qquad\qquad -1 < \bar{\chi} \le 1 \;.$$

(7)

Values of $\overline{\chi}>0$, $\overline{\chi}=0$ and $\overline{\chi}<0$ are an approximate measure of positive dependence, exact independence, and negative dependence in the tails, respectively. In particular, $\overline{\chi}$ resembles a correlation coefficient, and it is identical to the Pearson correlation coefficient under normality.

Poon et al's tail-dependence test is based on the $(\chi, \overline{\chi})$ pair, which makes it possible to characterize both the form and degree of extreme-value dependence. For asymptotically dependent variables, $\overline{\chi}=1$ and the degree of dependence is measured by $\chi>0$. For asymptotic independent variables, $\chi=0$ and the degree of dependence is measured by $\overline{\chi}$.

The above tail-dependence test rests on the fact that

$$\Pr(Z>z)= z^{-1/\zeta}L(z) \qquad \text{for } z>u,$$

(8)

for some high threshold u, where Z=min(S,T). Equation (8) shows that $\zeta$ is the tail index of the univariate random variable Z. Therefore, it can be computed by using the Hill estimator, constrained to the interval (0, 1]. Under the assumption of independent observations on Z, Poon et al. show that

$$\hat{\overline{\chi}} = 2\hat{\zeta}-1 = \frac{2}{n_u}\left(\sum_{j=1}^{n_u}\log\left(\frac{z_{(j)}}{u}\right)\right)-1 \qquad \text{Var}(\hat{\overline{\chi}}) = \frac{(\hat{\overline{\chi}}+1)^2}{n_u},$$

(9)

where $\hat{\overline{\chi}}$ is asymptotically normal.

The null hypothesis of asymptotic dependence (i.e., $\overline{\chi}=1$) is rejected if $\hat{\overline{\chi}}+1.96\sqrt{\text{Var}(\hat{\overline{\chi}})}<1$. In that case, we conclude that the two random variables are

asymptotically independent (i.e., $\chi=0$), and the degree of dependency is measured by $\overline{\chi}$. Otherwise, if the null hypothesis cannot be rejected, $\chi$ is estimated under the assumption that $\overline{\chi} = \xi = 1$, where $\hat{\chi} = \dfrac{un_u}{n}$ and $Var(\hat{\chi}) = \dfrac{un_u(n-n_u)}{n^3}$.

### 2.1.2 Threshold selection

In order to compute the Hill estimator of the tail index referred to above ($\zeta$), one has to choose an appropriate threshold (u). The simplest procedure is to plot the Hill estimator on u, and find such u for which it stabilizes (see, for instance, Tsay, 2001, chapter 7). In practice, however, in some cases such graphical procedure may not shed much light on the optimal threshold to be selected. Consequently, formal methods to choose u have been devised. A discussion on different adaptive-threshold selection can be found in Matthys and Beirlant (2000).

The authors distinguish two approaches to estimating the optimal threshold u. One consists of constructing an estimator for the asymptotic mean-squared error (AMSE) of the Hill estimator, and choosing the threshold that minimizes it. This approach includes a bootstrap method (e.g., Danielson, de Haan, Peng, and de Vries, 2001). The second approach directly derives an estimator for u, based on the representation of the AMSE of the Hill estimator. The exponential regression model—studied in detailed in Beirlant, Diercks, Goegebeur, and Matthys (1999), and further discussed in Matthys and Beirlant (2003)—falls into this class. Given that the exponential regression approach is both straightforward and computationally fast, it is our choice to find the optimal threshold. [3] We next describe the steps involved in this procedure.

---

[3] Matthys and Berlaint (2000) carry out simulation exercises under different distributional assumptions to compare the adaptive-threshold selection methods they discuss. They find that the exponential-regression

Feuerverger and Hall (1999) and Beirlant et al (1999) derive an exponential regression model for the log-spacings of upper statistics

$$j(\log(X_{n-j+1,n})-\log(X_{n-j,n})) \sim \left( \gamma + b_{n,k} \left( \frac{j}{k+1} \right)^{-\rho} \right) f_j \qquad 1 \leq j \leq k, \tag{10}$$

where $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$, $b_{n,k} = b\left( \dfrac{n+1}{k+1} \right)$, $1 \leq k \leq n-1$, $(f_1,\ f_2,\ \ldots, f_k)$ is a vector of independent standard exponential random variables, and $\rho \leq 0$ is a real constant.

If the threshold u is fixed at the $(k+1)^{th}$ largest observation, the Hill estimator can be rewritten as

$$H_{n,k} = \frac{1}{k} \sum_{j=1}^{k} j(\log(X_{n-j+1,n}) - \log(X_{n-j,n})). \tag{11}$$

The Hill estimator so expressed is the maximum-likelihood estimator of $\gamma$ in the reduced model

$$j(\log(X_{n-j+1,n}) - \log(X_{n-j,n})) \sim \gamma f_j \qquad 1 \leq j \leq k.$$

From the above, the AMSE of the Hill estimator is given by

$$\text{AMSE } H_{k,n} = \left( \frac{b_{n,k}}{1-\rho} \right)^2 + \frac{\gamma^2}{k}. \tag{12}$$

Therefore, the optimal threshold $k_n^{opt}$ is defined as the one that minimizes (12):

$$k_n^{opt} \equiv \arg\min_k (\text{AMSE } H_{k,n}) = \arg\min_k \left( \left( \frac{b_{n,k}}{1-\rho} \right)^2 + \frac{\gamma^2}{k} \right). \tag{13}$$

The algorithm for the exponential regression goes as follows:

---

method performs quite well, and that it even outperforms the bootstrap method for moderate sample sizes (e.g. 500).

- In expression (10), fix $\rho$ at $\rho_0 = -1$ and calculate least-squares estimates $\hat{\gamma}_k$ and $\hat{b}_{n,k}$ for each $k \in \{3,\ldots, n\}$.

- Determine $\overline{\text{AMSE H}_{k,n}} = \left(\dfrac{\hat{b}_{n,k}}{1 - \hat{\rho}_k}\right)^2 + \dfrac{\hat{\gamma}_k^2}{k}$ for $k \in \{3,\ldots, n\}$, with $\hat{\rho}_k \equiv \rho_0$. [4]

- Determine $\hat{k}_n^{\text{opt}} = \underset{3 \le k \le n}{\arg \min}\, (\overline{\text{AMSE H}_{k,n}})$ and estimate $\gamma$ by $H_{\hat{k}_n^{\text{opt}}}$.

The first step of the algorithm boils down to running a linear regression of

$$j(\log( X_{n-j+1,n}) - \log( X_{n-j,n}))$$ on a constant term and $\dfrac{j\,(n+1)}{(k+1)^2}$, for each $k \in \{3,\ldots, n\}$.

## 2.2 Copula analysis

### 2.2.1 Basic ideas

Copula function methodology has arisen as a new technique to measure the co-movement between financial markets. Copulas are uniform distributions which enable us to extract the dependence structure from the joint probability distribution function of a set of random variables and, at the same time, to separate the dependence structure from the univariate marginal behavior. Examples of recent applications of copulas in finance are Cherubini and Luciano (2002, 2003a, b), Embrechts, Lindskog and McNeil (2003), Giesecke (2004), Junker, Szimayer, and Wagner (2005), Pachenko (2005), and Rosenberg and Schuermann (2006). A thorough discussion of the use of copulas in finance is provided in the textbook by Cherubini, Luciano, and Vecchiato (2004). In addition, the survey article by Frees and Valdez (1998) provides an excellent background on the use of copulas in a more general context.

---

[4] Matthys and Beirlant (2000) point out that for many distributions the exponential-regression method works better, in MSE-sense, if the nuisance parameter $\rho$ is fixed at some value $\rho_0$ rather than estimated.

A copula is defined as a multivariate distribution function (df) F of random variables $X_1,..., X_n$ with standard uniform marginal cumulative distribution functions $F_1,..., F_n$ (margins). That is, $X_i \sim F_i$, i=1,..., n. Consequently, a copula satisfies the following properties (see Matteis 2001, section 2):

i)      $C(x_1,..., x_n)$ is increasing in each component $x_i$

ii)     $C(1,..,1,x_i,1,...,1)=x_i \ \forall \ i=1,...,n, \ x_i \in [0,1]$

iii)    For all $(a_1,..,a_n), (b_1,...,b_n) \in [0,1]^n$ with $a_i \leq b_i$

$$\sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+...+i_n} C(x_{1i},...,x_{ni_n}) \geq 0$$

with $x_{j1}=a_j$ and $x_{j2}=b_j \ \forall \ j \in \{1, ..., n\}$.

In general, let us consider an n x 1 random vector **X** with a joint df F and continuous margins $F_i$, which are not necessarily standard uniform.[5] Then

$$F(x_1,...,x_n) \qquad =Pr(X_1 \leq x_1, ..., X_n \leq x_n)$$

$$=Pr(F_1(X_1) \leq F_1(x_1), ..., F_n(X_n) \leq F_n(x_n))$$

$$=C(F_1(x_1),..., F_n(x_n)) \qquad\qquad\qquad (14)$$

Equation (14) shows that the joint df F can be described by the margins $F_1,.., F_n$ and the copula C. The latter captures the dependence structure among $X_1,..., X_n$. The existence of the function C is established by Sklar´s theorem (see Nelsen 1999, section 2.10). According to it, each multivariate distribution function with continuous margins has a unique copula representation.[6] Conversely, if C is a copula and $F_1,..., F_n$ are distribution functions, the function F given in (17) is a joint df with margins $F_1,.., F_n$.

---

[5] A well-known result in statistics establishes that if $X_i$ is a random variable with a continuous distribution function $F_i$, the random variable $F_i(X_i)$ is standard-uniformly distributed, i.e., $F_i(X_i) \sim U(0,1)$.
[6] If one or more margins exhibit discontinuities, the copula representation is not unique.

The density function of $X_1, ..., X_n$ in turn can be expressed in terms of the density copula and the marginal densities:

$$f(x_1, x_2, ..., x_n) = \left( \frac{\partial^n C(F_1(x_1), F_2(x_2), ..., F_n(x_n))}{\partial F_1(x_1) \partial F_2(x_2) ... \partial F_n(x_n)} \right) \frac{\partial F_1(x_1)}{\partial x_1} \frac{\partial F_2(x_2)}{\partial x_2} .... \frac{\partial F_n(x_n)}{\partial x_n}$$

$$= c(F_1(x_1), F_2(x_2)..., F_n(x_n)) \prod_{i=1}^{n} f_i(x_i) \qquad (15)$$

If n=2, we have a bivariate copula, which is defined on $I^2$=[0, 1] x [0, 1]:

$$F(x,y)=C(F_X(x), F_Y(y)) \equiv C(u, v)=Pr(U \leq u, V \leq v)$$

where $U=F_X(x)$ and $V=F_Y(y)$ are standard uniforms.

The joint density function of X and Y can be in turn expressed in terms of the copula density, $c(u,v) = \dfrac{\partial^2}{\partial u \partial v} C(u,v)$, and the corresponding marginal densities of X and Y according to equation (18). In this case, the empirical copula is given by[7]

$$\hat{C}\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} 1_{\{u_k \leq u(i), v_k \leq v(j)\}} \qquad i, j=1,2..,n \qquad (16)$$

where $u_{(1)} \leq u_{(2)} \leq ... \leq u_{(n)}$ and $v_{(1)} \leq v_{(2)} \leq ... \leq v_{(n)}$ are the order statistics.

Upper- and lower-tail dependence measures can be obtained as follows (see, for instance, Cherubini et al., section 3.1.5):

$$\lambda_u = \lim_{q \to 1^-} Pr(U > q \mid V > q) = \lim_{q \to 1^-} \frac{1 - 2q + C(q,q)}{1-q} \qquad (17a)$$

$$\lambda_l = \lim_{q \to 0^+} Pr(U < q \mid V < q) = \lim_{q \to 0^+} \frac{C(q,q)}{q} \qquad (17b)$$

---

[7] In general, the empirical copula can be obtained by $\hat{C}\left(\dfrac{i_1}{n}, \dfrac{i_2}{n}, ..., \dfrac{i_m}{n}\right) = \dfrac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{u_{1k} \leq u_1(i_1), u_{2k} \leq u_2(i_2), ... u_{mk} \leq u_m(i_m)\}}$ , $1 \leq j \leq m$, $i_j=1,2..,n$.

C is said to have upper-tail dependence if and only if $?_u \in (0, 1]$, and no upper-tail dependence if and only if $?_u = 0$. Similarly, C is said to have lower-tail dependence if and only if $?_l \in (0, 1]$, and no lower-tail dependence if and only if $?_l = 0$.

It is worth noticing that the coefficient $\chi$ in expression (6) can be generalized to

$$\chi = \lim_{q \to 1-} \Pr(U > q \mid V > q)$$

where U and V are the transformation of (X, Y) to uniform margins (see Coles, Heffernan, and Twan 1999). Therefore, under such transformation, $\chi$ coincides with $\lambda_u$.

One of the most frequently used copulas in the finance field is the Gaussian one. For the bivariate case, the Gaussian copula boils down to

$$C(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} dx \int_{-\infty}^{\Phi^{-1}(v)} dy \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)) \quad (18)$$

where $\Phi_\rho$ is the joint df with correlation coefficient $\rho$.

One characteristic of the Gaussian copula is that it does not exhibit either lower- or upper-tail dependence unless $\rho = 1$. That is to say,

$$\lambda_u = \lambda_l = \begin{cases} 0 & \text{iff } \rho < 1 \\ 1 & \text{iff } \rho = 1 \end{cases}.$$

However, assets returns may present extreme-value dependency in both tails.[8] Therefore, recent studies have focused on the Student's t-copula (e.g., Demarta and McNeil 2005; Mashal, Naldi, and Zeevi 2003). The bivariate t-copula is defined as

$$C(u, v) = \int_{-\infty}^{t_\upsilon^{-1}(u)} dx \int_{-\infty}^{t_\upsilon^{-1}(v)} dy \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{x^2 - 2\rho xy + y^2}{\upsilon(1-\rho^2)}\right)^{-\frac{\upsilon+2}{2}} = t_{\rho,\upsilon}(t_\upsilon^{-1}(u), t_\upsilon^{-1}(v))$$

(19)

---

[8] The Gumbel and Clayton copulas only allow for upper and lowe r tail dependence, respectively.

where $t_\upsilon(x) = \int_{-\infty}^{x} \dfrac{\Gamma((\upsilon+1)/2)}{\sqrt{\pi\upsilon}\,\Gamma(\upsilon/2)}\left(1 + \dfrac{z^2}{\upsilon}\right)^{-\frac{\upsilon+1}{2}} dz$ .

2.2.2   Maximum likelihood estimation of a bivariate t-copula

The Student's t copula density is given by (see, for instance, Cherubini, Luciano, and Vecchiato 2004, section 3.2.2)

$$c_{\rho,\upsilon}(u,v) = \rho^{-\frac{1}{2}} \frac{\Gamma\left(\dfrac{\upsilon+2}{2}\right)\Gamma\left(\dfrac{\upsilon}{2}\right)}{\Gamma\left(\dfrac{\upsilon+1}{2}\right)^2} \frac{\left(1 + \dfrac{\kappa_1^2 + \kappa_2^2 - 2\rho\kappa_1\kappa_2}{\upsilon(1-\rho^2)}\right)^{-(\upsilon+2)/2}}{\prod_{j=1}^{2}\left(1 + \dfrac{\kappa_j^2}{\upsilon}\right)^{-(\upsilon+2)/2}} \tag{20}$$

where $\kappa_1 = t_\upsilon^{-1}(u)$ and $\kappa_2 = t_\upsilon^{-1}(v)$ .

For a sample of n independent observations, estimates of $\rho$ and $\upsilon$ can be obtained by maximizing the log-likelihood function of the sample:

$$\log L = \sum_{i=1}^{n} \log(c_{\rho,\upsilon}(u_i, v_i)) \tag{21}$$

Given that both $\kappa_{1,i} = t_\upsilon^{-1}(u_i)$ and $\kappa_{2,i} = t_\upsilon^{-1}(v_i)$, i=1,..,n, depend on the unknown parameter $\upsilon$, we conduct a grid search over $\upsilon$ and maximize log L with respect to $\rho$, for every fixed value of $\upsilon$. We choose the paired combination that maximizes log L.[9]

2.2.3 Generation of random samples from Gaussian and t-Student copulas

A random sample from a Gaussian copula can be generated as follows (see Wang 1999 or Cherubini et al. 2004, section 6.1, for instance). Let $(X_1,.., X_n)$ be a set of correlated random variables with margins $F_{X_1},..., F_{X_n}$ and Kendall's tau $\tau_{ij}=\tau(X_i,X_j)$ or

---

[9] Our S-Plus code draws from that developed by Dean Fantazzini in GAUSS, which is freely available at http://economia.unipv.it/pagp/pagine_personali/dean/programs/t_copula_simul_est_new.

Spearman's rank correlation RankCorr($X_i$,$X_j$).[10] If the dependence structure can be adequately described by a normal copula, then the following algorithm can be used:

Step 1: Convert the given Kendall's tau or rank correlation coefficient to the pair-wise correlation coefficient used for normal random variables:

$$\rho_{ij} = \sin\left(\frac{\pi}{2}\tau_{ij}\right) = 2\sin\left(\frac{\pi}{6}\text{RankCorr}(X_i, X_j)\right)$$

and construct the lower triangular matrix **B**, such that **S**=**BB′** (i.e., Cholesky decomposition of **S**),where **S** is the matrix of correlation coefficients as computed above.

Step 2: Generate an n x 1 vector **y** of standard normal variables

Step 3: Let **z**=**By** and set $u_i$=$\Phi(z_i)$, i=1,,,, n.

Step 4: Set $x_i = F_{X_i}^{-1}(u_i)$, i=1,.., n.

Similarly, a sample of random variables, whose dependency can be modeled by a t-copula, can be generated as follows:

Steps 1 and 2: the same as above

Step 3: Simulate a random variable $\zeta$ from a $\chi_\upsilon^2$, independent of **y**

Step 4: Set **z**=**By**

Step 5: $\mathbf{w} = \sqrt{\upsilon/\zeta}\,\mathbf{z}$

Step 6: Set $u_i = t_\upsilon(w_i)$, i=1, 2, .., n, where $t_\upsilon(.)$ is the univariate t-distribution function.

Step 7: Set $x_i = F_{X_i}^{-1}(u_i)$, i=1,.., n.

---

[10] For the random variables X and Y, with continuous cdf F, Kendall's tau is defined as $\tau$=Pr{$(X_i-X_j)(Y_i-Y_j)>0$}−Pr{$(X_i-X_j)(Y_i-Y_j)<0$}. If C is the copula associated with F, it holds that $\tau = 4\iint_{I^2} C(u,v)c(u,v)dudv - 1$. In turn, Spearman's rank correlation coefficient is defined as the Pearson correlation coefficient between $F_X(X)$ and $F_Y(Y)$, where $F_X$ and $F_Y$ are the margins of X and Y, respectively. It holds that RankCorr(X,Y)= $12\iint_{I^2} C(u,v)dudv - 3$.

In order to model the margins $F_{X_i}$, i=1,,.n, we resort to a semi-parametric procedure discussed by Carmona (2004). Specifically, the tails of the distribution can be modeled by means of the generalized Pareto distribution, while the empirical distribution can be used to model the center of the distribution. That is, parametric and non-parametric approaches are used to model the tails and the center of the distribution, respectively.

## 3    Data and estimation results

Our data set comprises the following Morgan Stanley Capital Investment (MSCI) indices: U.S. Investable Market Value, U.S. Large Cap 300, U.S. Mid Cap 450, and U.S. Small Cap 1750. The data is measured at a daily frequency and covers 15 years: June 1992-June 2006. Some descriptive statistics are presented in Table 1. As previously found in other studies, returns exhibit excess kurtosis and negative skewness. All returns series are comparably volatile, as measured by their standard deviation and interquartile range.

Figure 1, panels (a) and (b) presents estimates of the tail index parameter of paired returns for large/mid cap and investable market value/small cap, based on empirical copulas. Specifically, $\hat{\lambda}_u = \lim_{q \to 1-} \frac{1 - 2q + \hat{C}(q,q)}{1 - q}$ and $\hat{\lambda}_l = \lim_{q \to 0+} \frac{\hat{C}(q,q)}{q}$, where the empirical copula is computed according to equation (16). For the large/mid cap pair, the estimates of upper and lower tail dependence are around 0.64-0.68 and 0.68-0.71, respectively, as shown in Panel (a) of Figure 1. The estimated tail dependency index parameters for the investable market value/small cap are by contrast slightly smaller: around 0.43-0.44 in the upper tail and 0.52-0.55 in the lower tail.

Table 2 reports the computation of Poon et al. ´s tail dependency test for both pairs. As we see in Panel (a), extreme-value dependency is not rejected for the large/mid cap pair in either tail, whereas for the investable market value/small cap, extreme-value dependency is accepted at the 5 percent significance level in the lower tail, but it is rejected in the upper tail at the 1 percent significance level. For the large/mid cap pair, the estimated tail index parameters are 0.69 and 0.67 in the lower and upper tails, respectively. These are fairly close to those reported above.

As shown in Panel (b) of Table 2, filtering the returns data by an AR(1)-GARCH(1,1) model leads to rejecting upper tail dependency in both paired returns series. The null hypothesis of lower tail dependency continues to be accepted for the large/mid cap pair, whereas it is now rejected at a lower significance level for the investable market value/small cap (i.e., 2 percent level).

Our next step consists of fitting a suitable copula to the data, based on the dependency tests just reported. We first focus on the raw data and then on the filtered data. Figure 2 shows the result of fitting normal and t-Student's copulas to the investable market value/small cap pairs. As a benchmark, the empirical copula is plotted along with each parametric model. As previously discussed, Poon et al.'s test suggests that there is tail independence in this pair. Therefore, a normal copula should be an appropriate choice (right-hand side panel of Figure 2). However, a t copula appears to be a better fit as it captures more accurately the dependence structure in the lower tail and in the center of the bivariate distribution. (The degrees of freedom and correlation coefficient are computed by the method of maximum likelihood, which was previously discussed).

Indeed, based on the Akaike, Schwarz, and Hannan-Quinn information criteria, the t-copula outperforms the normal copula.

We follow a similar procedure for the large/mid cap pair. Given that in this case, Poon et al.'s test does not reject lower and upper tail dependency, our choice is a t-copula. As a matter of comparison, we also fit a normal copula. As before, we conclude, based on the above three information criteria, that a t copula gives a better fit.

Further evidence on the goodness of fit of the t copula is provided by Figure 3, which depicts QQ-plots of actual and simulated returns for the four indices. In general, we see that the simulated data resembles the actual returns to a great extent.

As discussed earlier, after filtering the raw data, lower and upper tail dependency is rejected for the investable market value/small cap at the 5 percent level. Figure 4 shows that indeed the estimates of the tail index parameters for this pair are smaller than in the raw data, particularly so for the upper tail. Therefore, a normal copula should be in principle suitable to this filtered pair. Poon et. al.'s test suggests in turn that the large/mid cap pair only exhibits lower-tail dependency. Based on this fact, we fit a Clayton copula to the data, which allows for lower-tail dependency, and also a t-copula. Our estimation results show that for the latter pair, a t copula, with 6 degrees of freedom and a correlation coefficient of 0.9, mimics the dependency pattern of the data more closely than a Clayton copula. For the former pair, the normal copula appears to be the right choice.

In sum, for the large/mid cap pair, filtering does not reduce tail dependency considerably and a t copula continues to provide the best fit. The impact of filtering on tail dependency only translates into a greater estimate of the number of degrees of

freedom of the t copula.[11] For the second pair, however, filtering has an impact on our choice of a suitable copula. In particular, filtering washes away tail dependency, and a t copula becomes unsuitable. In the former case, Poon et al.'s test incorrectly suggests to pick a Clayton copula after filtering, whereas in the latter case it provides a proper guidance.

From our above results, it appears that Poon et al. test is biased towards rejecting the null hypothesis of tail dependency. In order to look into this issue, we carry out three Monte Carlo experiments, which are reported in Table 3. The first one consists of generating two returns series of 1,000 observations each from GARCH(1,1) models and assuming that the joint behavior of the two series is adequately modeled by a t copula. That is, by construction, our returns series exhibit dependency in both tails and the null hypothesis holds. The two other exercises consists of taking normal copulas with a moderate $\rho$ (=0.5) and a relatively high correlation coefficient (=0.8). In these two cases, by construction, the null hypothesis is false.[12] Each experiment is repeated 100 hundred times, and at each iteration Poon et al's test is computed for the lower and upper tail.

Panel (a) shows that Poon et al.´s test exhibits a severe size distortion. For instance, at the nominal size (i.e., significance level) of 1 percent, we reject the null hypothesis 58 and 55 percent of the time in the upper and lower tail, respectively. That is, the actual confidence level in each case is 42 and 45 percent, respectively, instead of 99 percent. Panels (b) and (c) illustrate the ability of the test to reject a false null hypothesis. For a small correlation coefficient of the normal copula, the power of the test approaches

---

[11] When the number of degrees of freedom is large enough, the t copula will approximate a normal copula, which does not display tail dependency unless the correlation coefficient equals 1.
[12] The normal copula only exhibits tail dependency when $\rho$=1, in which case $\lambda_l=\lambda_u=1$.

1. That is, the false null hypothesis is virtually always rejected. However, the power of the test decreases to a great extent for a higher correlation coefficient. For instance, when $\rho=0.8$, the power of the test is only 33 and 28 percent in the lower and upper tail, respectively, for a significance level of 1 percent.

Based on our findings, we conclude that copulas offer a reliable methodology to find a suitable functional form that describes accurately the dependence pattern of financial returns. By contrast, Poon et al's tail dependency test may be misleading as it is biased towards rejecting the null hypothesis of asymptotic dependence.

## 4    Conclusions

We discuss the choice of an optimal copula function of paired returns aimed at adequately capturing the co-movement between the two financial series. Our application focuses on daily data of four Morgan Stanley U.S. stock indices: U.S. Investable Market Value, U.S. Large Cap 300, U.S. Mid Cap 450, and U.S. Small Cap 1750, for the sample period June 1992-June 2006. Our estimation results show that a t-Student's copula, which allows for lower- and upper-tail dependency, works well in general, and that, filtering returns may have an impact on the choice of the most suitable copula.

We also computed Poon et al.'s dependency test to complement our analysis, and found that this can be sometimes misleading as a guidance to select a suitable copula. We further discussed this issue by means of Monte Carlo simulations, which showed that Poon et. al's test may exhibit size distortions and low power.

## References

Beirlant, Jan, Goedele Dierckx, Yuri Goegebeur, and Gunther Matthys (1999), "Tail index estimation and an exponential regression model", Extremes 2: 177-200.

Carmona, R., 2004, Statistical Analysis of Financial Data in Splus. Springer-Verlag, New York.

Cherubini, U., and E. Luciano (2002) "Bivariate option pricing with copulas." Applied Mathematical Finance 9(2), 69-85.

_____(2003a). "Pricing and hedging credit derivatives with copulas." Economic Notes 32(2), 219-242.

_____(2003b). "Pricing vulnerable options with copulas." Journal of Risk Finance 5(1), 27-39.

Cherubini, U., E. Luciano, and W. Vecchiato (2004). Copula methods in finance. Wiley.

Coles, Stuart, Janet Heffernan, and Jonathan Tawn (1999), "Dependence measures for extreme value analyses." Extremes 2: 339-365.

Coles, S. (2001), An Introduction to Statistical Modeling of Extreme Values. Springer series in statistics, Springer-Verlag London Limited.

De Matteis, R. (2001). Fitting copulas to data. Diploma thesis, Institute of Mathematics of the University of Zurich. www.creditriskresource.com/papers/paper_53.pdf.

Embrechts P., Klüppelberg, C. and Mikosch, T. (1997), Modelling Extremal Events for Insurance and Finance. Springer-Verlag Berlin Heidelberg.

Embrechts, P., F. Lindskog, and A. McNeil (2003). "Modeling dependence with copulas and applications to risk management." Handbook of Heavy Tailed Distribution in Finance, S.T. Rachev, editor, chapter 8, 331-384. JAI Press: Handbooks in Finance, Volume 1.

Frees, E., and E. Valdez (1998). "Understanding relationships using copulas." North American Actuarial Journal 2(1), 1-25.

Giesecke, K. (2004). "Correlated default with incomplete information." Journal of Banking & Finance 28(7), 1521-1545.

Junker, M., A. Szimayer, and N. Wagner (2005). "Nonlinear term structure dependence: Copula functions, empirics, and risk implications." Journal of Banking & Finance, In Press.

Mashal, R., M. Naldi, and A. Zeevi (2003). "Comparing the dependence structure of equity and asset returns" Risk 16, 82-87.

Matthys, Gunther, and Jan Beirlant (2000), "Adaptive threshold selection in tail index estimation", mimeo, Center for Statistics at the Catholic University Leuven.

Matthys, Gunther, and Jan Beirlant (2003), "Estimating the extreme value index and high quantile with exponential regression models", *Statistica Sinica* **13**: 853-880.

Nelsen, R. (1999). An introduction to copulas. Lecture Notes in Statistics 139. Springer-Verlag New York, Inc.

Pachenko, V. (2005). "Goodness-of-fit test of copulas." Physica A 355, 176-182.

Poon, S., M. Rockinger, and J. Tawn (2003), "Modeling extreme-value dependence in international stock markets", Statistica Sinica 13: 929-953.

Poon, S., M. Rockinger, and J. Tawn (2004), "Extreme value dependence in financial markets: Diagnostics, models, and financial implications", Review of Financial Studies 17: 581-610.

Wang, S. (1999). "Understanding relationships using copulas, Edward Frees and Emiliano Valdez, January 1998." North American Actuarial Journal 3(1), 137-141.

# Tables

**Table 1** Descriptive statistics of MSCI U.S. returns: June 1992-June 2006

| Statistic | U.S. Investable Market value | U.S. Large Cap 300 | U.S. Mid Cap 450 | U.S. Small Cap 1750 |
|---|---|---|---|---|
| Minimum | −0.060 | −0.072 | −0.083 | −0.063 |
| Maximum | 0.009 | 0.010 | 0.010 | 0.010 |
| 1$^{st}$ Q | −0.004 | −0.005 | −0.004 | −0.005 |
| 3$^{rd}$ Q | 0.005 | 0.005 | 0.006 | 0.006 |
| Mean | 0.000 | 0.000 | 0.000 | 0.000 |
| Std. Dev | 0.051 | 0.057 | 0.055 | 0.050 |
| Skewness | −0.183 | −0.080 | −0.302 | −0.272 |
| Kurtosis −3 | 3.974 | 4.324 | 4.458 | 2.994 |
| Observations | 3,674 | 3,674 | 3,674 | 3,674 |

Notes: Log-returns are daily

**Table 2** Extreme-value dependency test

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) Raw data | | | | | | | | | | | | | | | |
| | | | | Lower tail | | | | | | | Upper tail | | | | |
| Paired return | ρ | k* | $\overline{\chi}$ | s.e | t-test $\overline{\chi}=1$ | p-value | χ | s.e | k* | $\overline{\chi}$ | s.e | t-test $\overline{\chi}=1$ | p-value | χ | s.e |
| Large/mid cap | 0.91 | 191 | 0.92 | 0.14 | −0.61 | 0.27 | 0.69 | 0.05 | 250 | 0.96 | 0.12 | −0.31 | 0.38 | 0.67 | 0.04 |
| Value/small cap | 0.81 | 299 | 0.83 | 0.11 | −1.63 | 0.05 | 0.64 | 0.03 | 297 | 0.75 | 0.10 | −2.48 | 0.01 | 0.59 | 0.03 |
| (b) Filtered data | | | | | | | | | | | | | | | |
| Large/mid cap | 0.90 | 221 | 0.95 | 0.13 | −0.35 | 0.36 | 0.68 | 0.04 | 299 | 0.77 | 0.10 | −2.29 | 0.01 | 0.68 | 0.04 |
| Value/small cap | 0.80 | 231 | 0.76 | 0.12 | −2.1 | 0.02 | 0.64 | 0.04 | 299 | 0.66 | 0.10 | −3.5 | 0.00 | 0.55 | 0.03 |

**Table 3** Simulation of rejection rates of tail dependency test

| Percentage rejection rate of H$_0$: tail dependence | | |
|---|---|---|
| (a) Data generating process: t Student's copula (υ=5, ρ=0.5) | | |
| 1 % significance level | 5 % significance level | 10 % significance level |
| Lower tail 58 | 82 | 86 |
| Upper tail 55 | 80 | 90 |
| (b) Data generating process: Normal copula (ρ=0.5) | | |
| 1 % significance level | 5 % significance level | 10 % significance level |
| Lower tail 98 | 100 | 100 |
| Upper tail 98 | 100 | 100 |
| (c) Data generating process: Normal copula (ρ=0.8) | | |
| 1 % significance level | 5 % significance level | 10 % significance level |
| Lower tail 33 | 65 | 78 |
| Upper tail 28 | 65 | 82 |

Notes: (1) Individual return series of 1,000 observations each are generated from GARCH(1,1) processes, and marginal distribution functions are estimated according to Carmona (2004)'s semi-parametric procedure. (2) Results are obtained from 100 simulations.

**Figure 1**    Tail index parameters of raw returns computed from the empirical copulas

(a)

Tail index of the Large- and Mid-Cap pair



(b)

Tail index of the Market-Value and Small-Cap pair
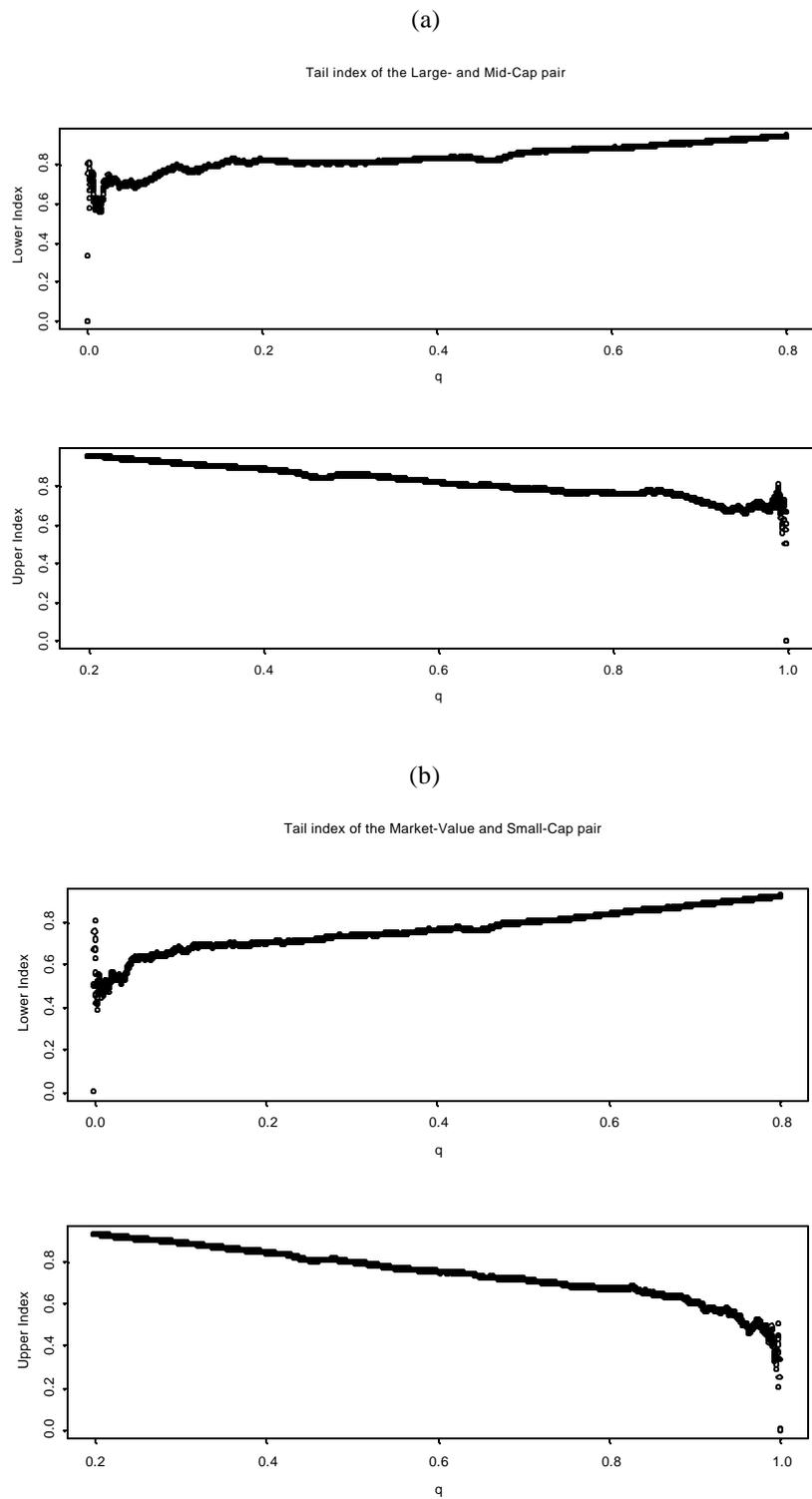
**Figure 2**     t-Student's and normal copulas fitted to the Investable Market Value & Small CAP 1750 pair



t-copula (v=4, rho=0.8) and empirical copula

normal copula (rho=0.8) and empirical copula
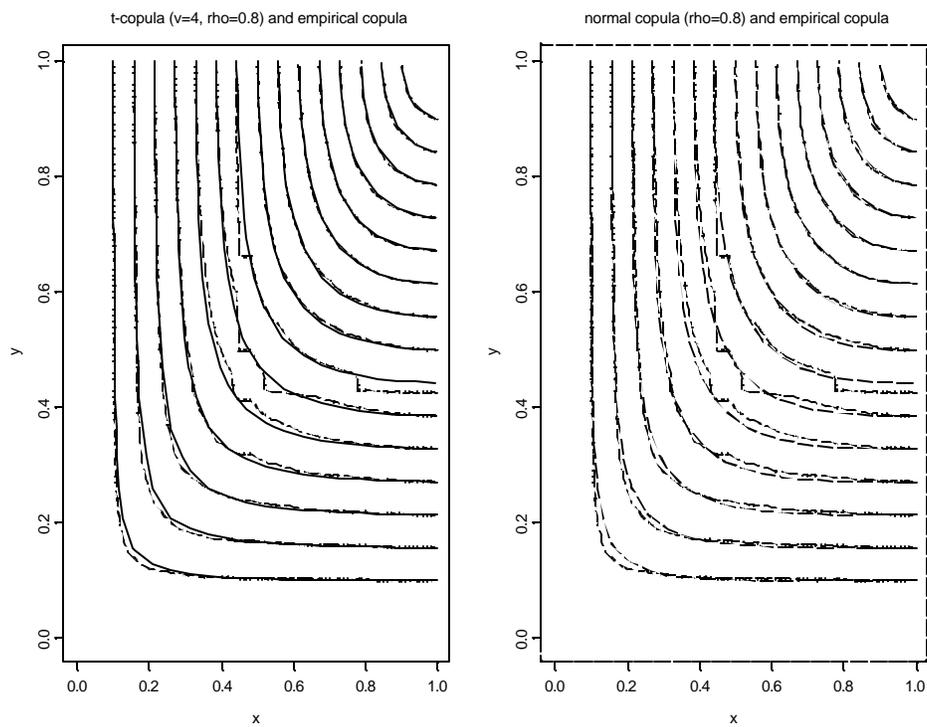
**Figure 3**           QQ-plot of actual and simulated returns based on a t-copula

(a)



(b)
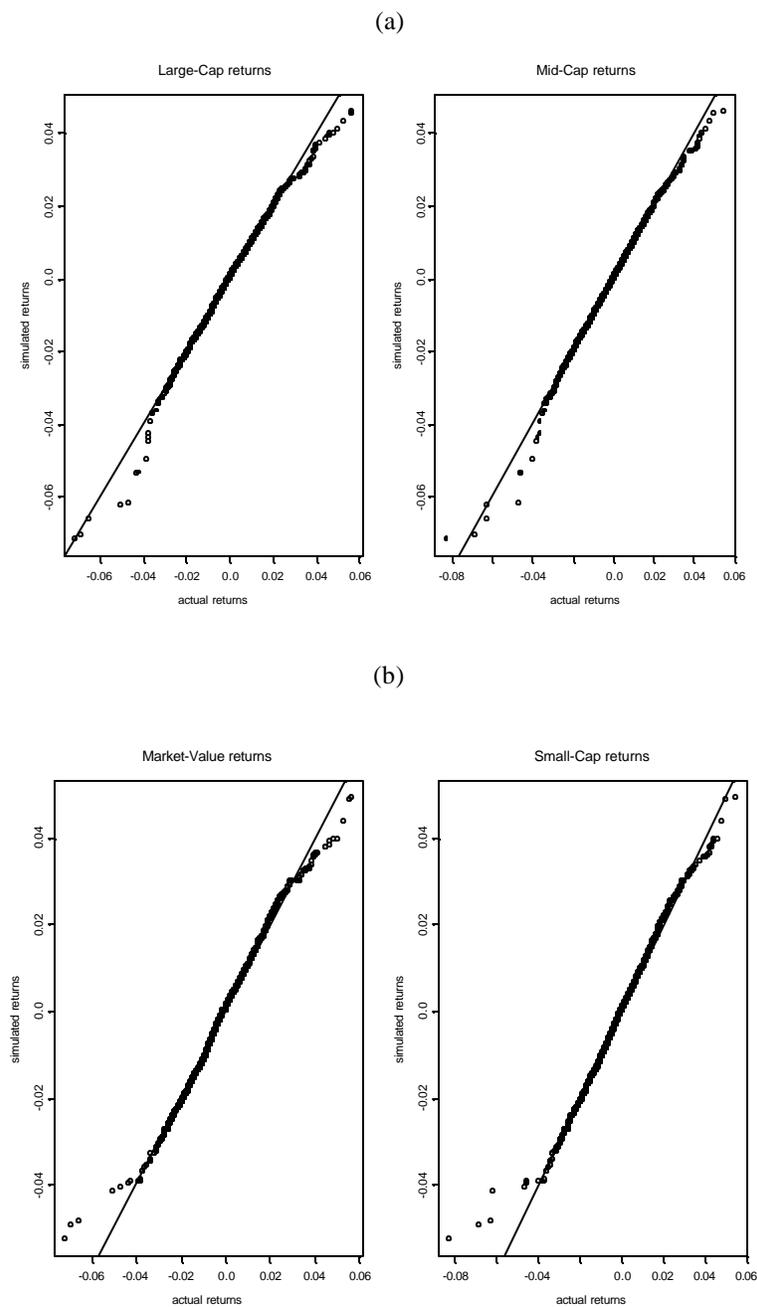
**Figure 4**    Tail index parameters computed from empirical copulas of filtered returns

(a)

Tail index of the Large- and Mid-Cap pair



(b)

Tail index of the Market-Value and Small-Cap pair
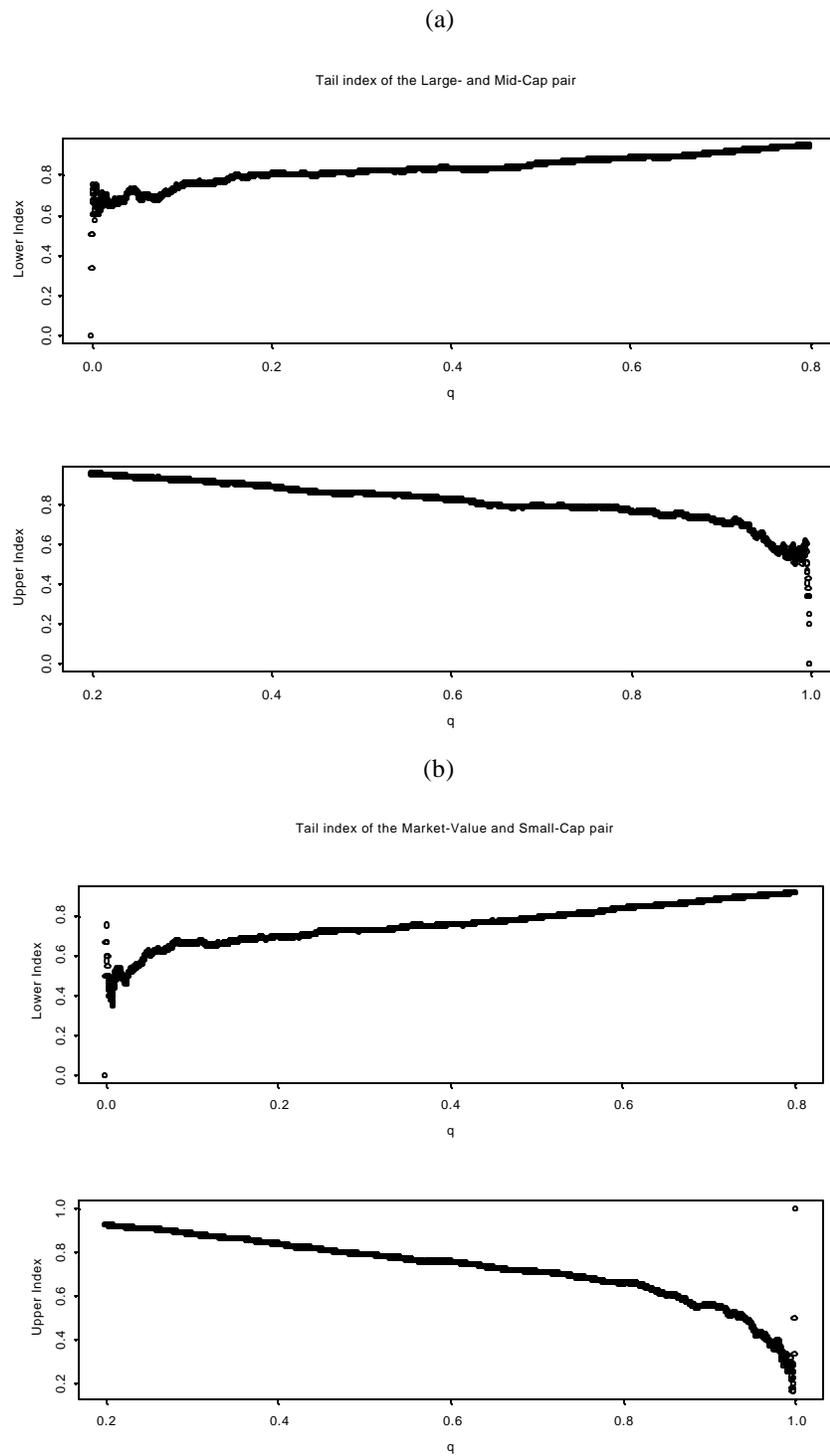
# Centro de Economía Aplicada
# Departamento de Ingeniería Industrial
# Universidad de Chile

## Serie Economía

## 2006

228. Copula-based measures of dependence structure in assets returns
Viviana Fernandez

227. Un Análisis Econométrico del Consumo Mundial de Celulosa
José Ignacio Sémbler, Patricio Meller y Joaquín Vial

226. The Old and the New Reform of Chile's Power Industry. (Por aparecer en el International Journal of Global Energy Issues (forthcoming 2007)).
M. Soledad Arellano

225. Socioeconomic status or noise? Tradeoffs in the generation of school quality information. (Por aparecer en el Journal of Development Economics).
Alejandra Mizala, Pilar Romaguera y Miguel Urquiola.

224. Mergers and CEO power
Felipe Balmaceda

123. Task-Specific Training and Job Design.
Felipe Balmaceda

122. Performance of an economy with credit constraints, bankruptcy and labor inflexibility
Felipe Balmaceda y Ronald Fischer

121. Renegotiation without Holdup: Anticipating spending and infrastructure concessions
Eduardo Engel, Ronald Fischer y Alexander Galetovic

220. Using School Scholarships to Estimate the Effect of Government Subsidized Private Education on Academic Achievement in Chile
Priyanka Anand, Alejandra Mizala y Andrea Repetto

219. Portfolio management implications of volatility shifts: Evidence from simulated data
Viviana Fernandez y Brian M Lucey

218. Micro Efficiency and Aggregate Growth in Chile
Raphael Bergoeing y Andrea Repetto

# 2005

217. Asimetrías en la Respuesta de los Precios de la Gasolina en Chile
Felipe Balmaceda y Paula Soruco

216. Sunk Prices and Salesforce Competition
Alejandro Corvalán y Pablo Serra

215. Stock Markets Turmoil: Worldwide Effects of Middle East Conflicts
Viviana Fernández

214. The Competitive Role of the Transmission System in Price-regulated Power Industries
M. Soledad Arellano y Pablo Serra

213. La Productividad Científica de Economía y Administración en Chile. Un Análisis Comparativo (Documento de Trabajo Nº 301. Instituto de Economía, Pontificia Universidad Católica de Chile)
Claudia Contreras, Gonzalo Edwards y Alejandra Mizala

212. Urban Air Quality and Human Health in Latin America and the Caribbean
Luis A. Cifuentes, Alan J. Krupnick, Raúl O'Ryan y Michael A. Toman

211. A Cge Model for Environmental and Trade Policy Analysis in Chile: Case Study for Fuel Tax Increases
Raúl O'Ryan, Carlos J. de Miguel y Sebastian Millar

210. El Mercado Laboral en Chile Nuevos Temas y Desafíos
Jaime Gatica y Pilar Romaguera

209. Privatizing Highways in The United States
Eduardo Engel, Ronald Fischer y Alexander Galetovic

208. Market Power in Price-Regulated Power Industries
M. Soledad Arellano y Pablo Serra

207. Market Reforms and Efficiency Gains in Chile
Raphael Bergoeing, Andrés Hernando y Andrea Repetto

206. The Effects on Firm Borrowing Costs of Bank M&As
Fabián Duarte, Andrea Repetto y Rodrigo O. Valdés

190. Trade Liberalization in Latin America: The Case of Chile
    Ronald Fischer

189. Time-Scale Decomposition of Price Transmission in International Markets
    Viviana Fernández

188. Slow Recoveries. (Por aparecer en Journal of Development Economics)
    Raphael Bergoeing, Norman Loayza y Andrea Repetto

187. Market Power in Mixed Hydro-Thermal Electric Systems
    M. Soledad Arellano

186. Efectos de la privatización de servicios públicos en Chile: Casos sanitario, electricidad y telecomunicaciones
    Ronald Fischer y Pablo Serra

185. A Hierarchical Model for Studying Equity and Achievement in the Chilean School Choice System
    Alejandra Mizala, Pilar Romaguera y Carolina Ostoic

184. Innovaciones en Productividad y Dinámica de Plantas. (Revista de Análisis Económico, 18(2), pp. 3-32, 2003)
    Raphael Bergoeing y Facundo Piguillem

183. The Dynamics of Earnings in Chile
    Cristóbal Huneeus y Andrea Repetto

182. Monopoly Regulation, Chilean Style: The Efficient-Firm Standard in Theory and Practice
    Álvaro Bustos y Alexander Galetovic

181. Vertical Mergers and Competition with a Regulated Bottleneck Monopoly
    Alexander Galetovic y Ricardo Sanhueza

180. Crecimiento Económico Regional en Chile: ¿Convergencia?
    Rodrigo Díaz y Patricio Meller

179. Incentives versus Synergies in Markets for Talent
    Bharat N. Anand, Alexander Galetovic y Alvaro Stein

178. Why is Manufacturing Trade Rising Even as Manufacturing Output is Falling?. (Por Aparecer en American Economic Review, Papers and Proceedings)
    Raphael Bergoeing, Tim Kehoe, Vanessa Strauss-Kahn and Kei-Mu Yi

177. Transmisión eléctrica y la "ley corta": por qué licitar es (mucho) mejor que regular
    Alexander Galetovic y Juan Ricardo Inostroza

176. Soft Budgets and Highway Franchising
    Eduardo Engel, Ronald Fischer and Alexander Galetovic

* Para ver listado de números anteriores ir a http://www.cea-uchile.cl/.