(Group) Strategy-proofness and stability in many-to-many matching markets *

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Abstract

We study strategy-proofness in many-to-many matching markets. We prove that when firms have acyclical preferences over workers and both firms and workers have responsive preferences, the worker-optimal stable mechanism is group strategy-proof and Pareto optimal. Absent any assumption on workers' preferences, an Adjusted Serial Dictatorship among workers is stable, group strategy-proof and Pareto optimal for workers. In both cases, the set of stable matchings is a singleton. We show that acyclicity is the minimal condition guaranteeing the existence of stable and strategy-proof mechanisms in many-to-many matching markets.

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1 Introduction

We explore the possibility of designing stable and strategy-proof mechanisms in many-to-many markets. In particular, we provide a necessary and sufficient condition for the worker-optimal stable mechanism to be strategy-proof.

The canonical example of a many-to-many market is the specialty training followed by junior doctors in the UK, where doctors had to arrange separate medical residencies and training positions with several hospitals (Roth, 1991). Nevertheless, all labor markets in which workers are allowed to work part-time fit our model, for example the market for part-time lecturers or the market for consultants. Furthermore, there are industries in which the relationship between agents has a many-to-many structure. Relevant examples are the non-exclusive dealings between car producers and parts suppliers or between health insurers and health care providers. Finally, many-to-many markets are useful for understanding multi-unit assignment problems such as the course allocation problem (see Budish and Cantillon, 2012; Sönmez and Ünver, 2010) or the problem of the assignment of landing slots (see Schummer and Abizada, 2017; Schummer and Vohra, 2013).[1]

The interactions between agents in many-to-many markets are complex. Our tools for addressing this complexity are limited by the fact that no stable and strategy-proof mechanism exists, even for the agents on one side of the market. This negative result is challenging because stability and strategy-proofness are central concerns in market design. Indeed, theoretical and empirical findings suggest that the markets that achieve stable outcomes are more successful than the markets that do not achieve stable outcomes (see Roth and Sotomayor, 1990; Abdulkadiroğlu and Sönmez, 2013). Moreover, in the school assignment model and the course allocation problem, stability embodies a notion of fairness, as it eliminates justified envy, that is, situa-

[1] In the case of multi-unit assignment problems and school choice the preferences of the firms over individual workers should be interpreted as priorities.

tions in which an agent prefers to receive another assignment over one of her assignments and has a higher priority at the preferred assignment (see Balinski and Sönmez, 1999; Sönmez, and Abdulkadiroğlu, 2003). With respect to strategy-proofness, it prevents the agents from needing to strategize, which is relevant in markets in which agents have little information and might differ in their sophistication.

To address the incompatibility between stability and strategy-proofness, the literature follows two different approaches. The first is to design approximate solutions such as the "approximate competitive equilibrium from equal incomes" introduced by Budish (2011) (see also Budish et al, 2016) for the combinatorial assignment problem. The second approach is to restrict the preference (or priority) domain to guarantee the existence of a stable and strategy-proof mechanism (see Jiao and Tian, 2017). In the context of the multi-unit assignment problem, this amounts to designing priorities that permit the existence of stable and strategy-proof mechanisms. We follow the latter approach.

We assume that the preferences of the firms over individual workers are acyclical. A cycle in the preferences of firms occurs when we can form an alternating list of firms and workers "on a circle" such that every firm prefers the worker on its clockwise side to the worker on its counterclockwise side and finds both acceptable. We focus our attention on the worker-optimal stable mechanism. In many-to-one matching markets, it is weakly Pareto optimal and strategy-proof for workers, but in many-to-many matching markets it is not weakly Pareto optimal nor strategy-proof for workers, even if we restrict the preferences of the agents to be responsive (see Roth and Sotomayor, 1990). We show that, if the preferences of the firms do not have cycles, the worker-optimal stable mechanism is group strategy-proof for workers and Pareto optimal for workers. Furthermore, if the preferences of the firms are acyclical and responsive, the set of stable matchings is a singleton. These results do not depend on any assumption on the preferences of the workers.

To prove the results, we show that, if the preferences of the firms are acyclical, the unique stable matching can be implemented through a procedure that we call Adjusted Serial Dictatorship. In an Adjusted Serial Dictatorship, each worker, at her turn, selects her favorite firms among those that she is acceptable to and still have vacant positions.

Next, we study the general group strategy-proofness of the worker-optimal stable matching. Since the worker-optimal stable matching does not exist under general preference restrictions, we also assume that the preferences of the workers are responsive. We prove that, when the preferences of the firms are acyclical, the worker-optimal stable mechanism is group strategy-proof.

Finally, we prove that acyclicity is the minimal condition guaranteeing the existence of stable and strategy-proof mechanisms in many-to-many matching markets, even if we restrict the preferences of the workers to be responsive.

To the best of our knowledge, this is the first paper that studies the properties of acyclical preferences in many-to-many matching markets. Acyclical preferences have been extensively analyzed in one-to-one and many-to-one matching markets. Ergin (2002) introduces a weaker notion of acyclicity, and shows that the worker-optimal stable matching is efficient and group strategy-proof if and only if the preferences of the firms are acyclical. Haeringer and Klijn (2009) show that Ergin's acyclicity is a necessary and sufficient condition for Nash implementation of the stable correspondence through the worker-optimal stable mechanism. Kesten (2012) and Romero-Medina and Triossi (2013a) study the impact of acyclicity on capacity manipulation.

The concept of acyclicity that we use coincides with that introduced in Romero-Medina and Triossi (2013b) for one-to-one matching markets. We extend their contribution to a many-to-many setting and recover the results of Dubins and Freedman (1981) on the strategy-proofness of the worker-optimal stable mechanism in a restricted preference domain. Our results also complement those in Jiao and Tian (2017). These authors prove that the worker-optimal stable matching is group strategy-proof for workers if preferences satisfy the extended max-min criterion and a quota saturability condition. This preference domain reflects a high degree of ambiguity aversion in agents. Instead, we assume that preferences are responsive.[2]

In contrast to Sönmez and Ünver (2010) and Budish and Cantillon (2013), most of our results do not depend on any assumption on the preferences of the workers. Thus, they imply that a designer choosing acyclical priorities should not worry about collusion among workers or about complementarities in their preferences. This last feature is particularly relevant in multi-unit assignment problems such as course allocation. In this case, complementarities among courses, peer effects and restrictions on the number of courses in which a student can enroll make the assumption of responsive preferences rather restrictive. For example, as observed in Sönmez and Ünver (2010), responsiveness does not account for the common cases in which courses are available in different sections, or have conflicting schedules. Moreover, students could be inclined to specialize in a variety of areas, but might not be interested in simultaneously attending courses in more than one area.

The concept of group strategy-proofness that we study is stronger than that considered by Dubins and Freeman (1981) and Jiao and Tian (2017). For an allocation to be group strategy-proof, they require that no coalition of workers can manipulate the mechanism, making every member of the coalition strictly better off. This is a concept that has been sometimes called group incentive compatibility (see Hatflied and Kojima, 2009). We use the concept of group strategy-proofness employed in Ergin (2002). For an allocation to be group strategy-proof, we require that no coalition of agents can manipulate the mechanism in a way that makes every member of the coalition weakly better off and at least one agent strictly better off.

Our results are related to Ehlers and Klaus (2003). In the setup of a multiple assignment problem, they characterize "sequential dictatorships" and serial dictatorships in terms of efficiency, group strategy-proofness and resource

[2]The two domains are unrelated.

monotonicity. Their results do not apply to our setting because their market is unilateral, meaning that there are no priority rights over objects.

The paper is organized as follows. Section 2 introduces the model and notation. Section 3 presents the results. Section 4 concludes.

2 The Model

There are two disjoint and finite sets of agents. Let F, $|F| \geq 2$ denote the set of firms and let W, $|W| \geq 3$ denote the set of workers. Let $V = F \cup W$. Generic agents are denoted by v, and generic firms and workers are denoted by f and w, respectively. Every firm f has a strict, complete and transitive preference relation P_f over 2^W. Moreover, every worker $w \in W$ has a strict, complete and transitive preference relation P_w over 2^F. For $v \in V$, R_v denotes weak preferences. For every $V' \subseteq V$ set $P_{V'} = (P_v)_{v \in V'}$ and $P_{-V'} = (P_v)_{v \in V \setminus V'}$. For every $v \in V$ set $P_{-v} = P_{-\{v\}}$. Let $f \in F$, and let $W' \subseteq W$. The choice set from W', $C_f(W', P_f)$ is f's favorite subset of W'; formally,

$$C_f(W', P_f) = A \iff A \subseteq W', AR_f A' \text{ for every } A' \subseteq W'.$$

We define the choice sets of workers similarly. For every $F' \subseteq F$ let

$$C_w(F', P_w) = B \iff B \subseteq F', BR_w B' \text{ for every } B' \subseteq F'.$$

When there is no ambiguity about preferences, we will write $C_f(W')$ and $C_w(F')$ for $C_f(W', P_f)$ and $C_w(F', P_w)$, respectively.

In our model, each firm can hire a set of workers and each worker can work for more than one firm. A matching is an assignment of workers to firms. Formally, a *matching* is a function $\mu : V \to 2^V$ such that, for all $f \in F$ and all $v \in V$,

1. $\mu(f) \in 2^W$;
2. $\mu(w) \in 2^F$; and
3. $w \in \mu(f)$ if and only if $f \in \mu(w)$.

We say that a matching μ is *individually rational* if, for every $v \in V$, $C_v(\mu(v)) = \mu(v)$. In other words, a matching is individually rational if no firm is willing to fire any of its employees and no worker is willing to quit any firm for which she is working. Thus, individual rationality captures the idea that hiring is voluntary.

We say that a matching μ is *blocked* by a pair $(f, w) \in F \times W$ if $f \notin \mu(w)$ and

1. $f \in C_w(\mu(w) \cup \{f\})$;
2. $w \in C_f(\mu(wf) \cup \{w\})$.

In words, a firm-worker pair (f, w) blocks a matching μ if worker w is not employed at f, but she would like to join f, eventually leaving some of her current jobs, and f would like to hire w, eventually firing some of its current employees. Finally, A matching μ is (pairwise) *stable* if it is individually rational and there exists no pair blocking it.

The set of stable matchings might be empty. For this reason, the literature has focused on restrictions on agents' preferences that guarantee that the set of stable matchings is non-empty. We assume that every firm f has underlying preferences over individual workers, formally a strict, complete and transitive preference relation, $\succ_f$ over $W \cup \{\emptyset\}$. If $w \succ_f \emptyset$, we say that worker w is acceptable to f. For every $f \in F$, $A(f, \succ_f)$ denotes the set of agents acceptable to f. When there is no ambiguity about $\succ_f$, we write $A(f)$ for $A(f, \succ_f)$. Furthermore, every firm $f \in F$ has a quota q_f which is the maximum number of workers that it can hire. Let $q = (q_f)_{f \in F}$ be the vector of quotas. We say that the preferences of firm $f \in F$, P_f over 2^W, are *responsive* to $\succ_f$, with quota q_f if, for all $W' \subseteq W$ and for all $w, w' \in W \backslash W'$, the following hold:

1. if $|W'| < q_f$, $w \succ_f w'$ if and only if $W' \cup \{w\} P_f W' \cup \{w'\}$;
2. if $|W'| < q_f$, $w \succ_f \emptyset$ if and only if $W' \cup \{w\} P_f W'$; and
3. if $|W'| > q_f$, then $\emptyset P_f W'$.

In words, firm f has responsive preferences if, for any two assignments that differ in only one worker, it prefers the assignment containing the more preferred worker. Moreover, the assignments with more workers than the firm's quota are not acceptable. Responsive preferences for workers can be defined similarly. Throughout the paper, we assume that the preferences of every firm f, P_f are responsive to $\succ_f$. When there is no ambiguity about $\succ_f$ we will simply say that P_f is responsive. Responsive preferences for workers can be defined similarly. A (many-to-many) matching market is denoted by $M = (F, W, P, q)$, where $P = (P_v)_{v \in V}$.

If all agents have responsive preferences the set of stable matchings is nonempty and is a complete distributive lattice (see Alkan, 1999). Furthermore, there exists a matching that is Pareto superior for workers to all other stable matchings, called *worker-optimal stable matching* and denoted by $\mu^W(P, q)$. Formally, for every matching that is stable for market (F, W, P, q), $\mu^W(P, q)(w) R_w \mu(w)$ for all $w \in W$.

A *cycle* (of length $T+1$) in $\succ_F$ is given by distinct workers $w_0, w_1, ..., w_T \in W$ and distinct firms $f_0, f_1, ..., f_T \in F$ such that

1. $w_T \succ_{f_T} w_{T-1} \succ_{f_{T-1}}, ..., \succ_{f_2} w_1 \succ_{f_1} w_0 \succ_{f_0} w_T$;
2. for every t, $0 \leq t \leq T$, $w_t \in A\left(f_{t+1}, \succ_{f_{t+1}}\right) \cap A\left(f_t, \succ_{f_t}\right)$,where $w_{T+1} = w_0$.

Let us assume that a cycle exists. If every worker w_{t-1} is initially assigned to firm f_t, every firm is willing to exchange its assigned worker with its successor w_t.

A preference profile on individual workers $\succ_F$ is *acyclical* if it has no cycles. We will assume that the firms' preference profile is acyclical.

A mechanism φ is a function that associates a matching to every preference profile P within a certain domain $\mathcal{P}$ and quota vector q. A mechanism φ is *stable* if $\varphi(P, q)$ is stable for all P and all q. The worker-optimal stable mechanism defined by $(P, q) \mapsto \mu^W(P, q)$ is an example of stable mechanism on the domain of responsive preferences. A mechanism φ is *Pareto optimal* if, for every P, there exists no individually rational matching μ, such that $\mu(v) R_v \varphi(P, q)(v)$ for every $v \in V$ and $\mu(v) P_v \varphi(P, q)(v)$ for at least one v. In words, a mechanism is Pareto optimal if it implements matchings for which there is no alternative individually rational matching that is weakly preferred by all agents and strongly preferred by at least one agent. A mechanism φ is *Pareto optimal* for workers if, for every P, there exists no individually rational matching μ, such that $\mu(w) R_w \varphi(P, q)(w)$ for every $w \in W$ and $\mu(w) P_w \varphi(P, q)(w)$ for at least one w. In words, a mechanism is Pareto optimal for workers if it implements matchings for which there exists no alternative individually rational matching that is weakly preferred by all other workers and strongly preferred by at least one worker. A mechanism φ is *strategy-proof* for workers if, for every $w \in W$, $\varphi(P, q)(w) R_w \varphi(P'_w, P_{-w}, q)(w)$ for every P, P'_v. In words, a mechanism is strategy-proof for workers if, reporting her true preference function is a (weakly) dominant strategy for every worker. A mechanism φ is *group strategy-proof* if there does not exist a nonempty set of agents, $V' \subset V$, P and $P'_{V'} = (P'_v)_{v \in V'}$ such that $\varphi(P'_{V'}, P_{V \setminus V'}, q)(v) R_v \varphi(P, q)(v)$ for every $v \in V'$ and $\varphi(P'_{V'}, P_{V \setminus V'})(v') P_{v'} \varphi(P)(v')$ for some $v' \in V'$. The mechanism φ is group strategy-proof if no subset of agents can benefit by jointly misrepresenting their preferences. Group strategy-proofness is a stronger requirement than group incentive compatibility. The latter requires that there exists no coalition of agents who can be strictly better off by misrepresenting their preferences (see Roth and Sotomayor, 1990; Hatfield and Kojima, 2009). Jiao and Tian's (2017) concept of group strategy-proofness coincides with group incentive compatibility. Furthermore, they restrict their attention to group deviations by workers. In many-to-many matching markets, if the preferences of both workers and firms are responsive, no stable mechanism is strategy-proof for workers, and no stable mechanism is Pareto optimal for workers; in particular, it is not group strategy-proof. In the next section, we show that the assumption of acyclical preferences is a necessary and sufficient condition to overcome the incompatibility of strategy-proofness, stability and Pareto optimality.

3 (Group) Strategy-proofness and uniqueness

In this section we present our results. We first prove that when the firms have acyclical preferences over individual workers, there exists an underlying order w_1,w_2,...,$w_{|W|}$ on the set of workers that is able to sustain a stable matching through an Adjusted Serial Dictatorship.

Assume that $\succ_F$ is acyclical, and define the following order on W. Let $w_1 \in W$ be a worker who is never ranked below first place by any firm to which she is acceptable. Formally, let w_1 be such that there exist no $w \in W$ and $f \in F$ with $w \succ_f w_1 \succ_f \emptyset$. Such a w_1 exists because $\succ_F$ is acyclical. For $0 \leq t \leq |W| - 1$, let w_{t+1} be a worker who is never ranked belowworkers different from $w_1, w_1, ..., w_t$ by any firm to which she is acceptable. Formally, let $w_{t+1} \in W$ be such that there exist no $w \in W \setminus \{w_1, w_1, ..., w_t\}$ and $f \in F$ such that $w \succ_f w_{t+1} \succ_f \emptyset$. Such a w_{t+1} exists because $\succ_F$ is acyclical. In general, the election of w_t is not unique, for every t, $1 \leq t \leq |W| - 1$ and thus, the procedure defines a family of orders on W. Proposition 2 below implies that all such orders generate the same stable matching and thus they are equivalent.

Next, we define the *Adjusted Serial Dictatorship* by letting each worker choose among the firms that she is acceptable to and still have vacant positions according to the order w_1,w_2,...,$w_{|W|}$, as defined above.

Let $A_1(P,q) = \{f \mid w_1 \in A(f)\}$, be the set of firms to which worker w_1 is acceptable. Define $\mu(P,q)(w_1) = C_{w_1}(A_1(P,q))$. For all t, $1 \leq t \leq |W|-1$, let $A_{t+1}(P,q) = \left\{ f \mid w_{t+1} \in A(f), \left| \bigcup_{s<t, f\in\mu(P,q)(w_s)} \{w_s\} \right| < q_f \right\}$, be the set of firms that worker w_{t+1} is acceptable to and that have vacant positions. Define $\mu(P,q)(w_{t+1}) = C_{w_{t+1}}(A_{t+1}(P,q))$. When there is no ambiguity about P or about q we will write μ instead than $\mu(P,q)$ and A_s instead of $A_s(P,q)$, for all s, $1 \leq s \leq |W|$.

First, we prove that matching μ, the outcome of the Adjusted Serial Dictatorship, is stable.

Lemma 1 *Let $M = (F,W,P,q)$ be a matching market and let $\succ_F$ be acyclical. The matching $\mu = \mu(P,q)$ is stable in the market (F,W,P,q).*

Proof. The definition of μ implies that it is individually rational. We prove the rest of the claim by contradiction. Assume that s and $f \in F$ exist such that (f, w_s) blocks μ. We have $\mu(w_s) = C_{w_s}(A_s)$. First, assume that $|\mu(f)| < q_f$. Then, $f \in A_s$, yielding a contradiction. Second, consider the case in which $|\mu(f)| = q_f$. Because (f, w_s) blocks μ, $w_s P_f w$ for some $w \in \mu(f)$. From the definition of the sequence $w_1, w_2, ..., w_{|W|}$, it follows that $w = w_l$ for some $l > s$. Thus, $f \in A_s$, yielding a contradiction. ■

Notice that Lemma 1 implies the existence of a stable matching whenever the preferences of the firms are responsive and acyclical, independent of the preferences of the workers.

Corollary 1 *Let $M = (F,W,P,q)$ be a matching market and let $\succ_F$ be acyclical. A stable matching exists in the market $M = (F,W,P,q)$.*

Moreover, the mechanism $\mu(P,q)$ is Pareto optimal for workers.

Proposition 1 *Let $M = (F, W, P, q)$ be a matching market, and let $\succ_F$ be acyclical. Then, matching $\mu = \mu(P, q)$ is Pareto optimal for workers.*

Proof. We prove the claim by contradiction. Assume that there exists an individually rational matching ν such that $\nu(w) R_w \mu(w)$ for every $w \in W$ and $\nu(w') P_{w'} \mu(w')$ for some $w' \in W$. Let $w_s \in W'$ be the maximal worker who benefits from matching ν. Formally, let s be the minimal integer such that $\nu(w_s) P_{w_s} \mu(w_s)$. All workers with an index lower than s are matched to the same firms under ν and under μ, and ν is individually rational; thus, $\nu(w_s) \subseteq A_s$. It follows that $\mu(P, q)(w_s) = C_{w_s}(A_s) R_s \nu(w_s)$, which yields a contradiction. ■

Next, we show that the set of stable matchings is a singleton whenever the preferences of the firms over individual workers are acyclical.

Proposition 2 *Let $M = (F, W, P, q)$ be a market and let $\succ_F$ be acyclical. Then, the set of stable matchings of M is a singleton.*

Proof. The proof of the claim is by contradiction. Let $\mu = \mu(P, q)$. Assume that $\nu \neq \mu$ is a stable matching. Let s be the minimal index such that $\mu(w_s) \neq \nu(w_s)$. Since ν is individually rational, then $\nu(w_s) \subseteq A_s$. It follows that $\mu(w_s) P_{w_s} \nu(w_s)$. Thus, there exists $f \in \mu(w_s) \setminus \nu(w_s)$. The minimality of s implies that either $|\nu(f)| < q_f$ or that there exists $t > s$ with $w_t \in \nu(f)$. Then (f, w_s) blocks ν yielding a contradiction. ■

To study the resistance of the worker-optimal stable matching to coalitional deviations, we have to restrict the domain of preferences. Indeed, the worker-optimal stable matching is not, in general, well defined, without restrictions on the preferences of the agents. More precisely, even if P_F is a profile of acyclical preferences for firms and P'_f is a profile of responsive preferences for firm f, the profile $P'_F = \left(P'_f, P_{F \setminus \{f\}}\right)$ is not, in general, acyclical. Thus, without any additional assumption on the preferences of the workers, a worker-optimal stable matching $\mu^W\left(P'_f, P_{-f}\right)$ might fail to exist. Furthermore, the

Adjusted Serial Dictatorship is not well defined without acyclicity. Thus, we assume that the preferences of the workers are responsive and prove our main result: no coalition of agents can benefit from preference manipulation if the worker-optimal stable mechanism $\mu^W(P,q)$ is used. We base our proof on the relationship between $\mu^W(P,q)$ and the sequential mechanism $\mu(P,q)$ defined above.

Theorem 1 *Let $M=(F,W,P,q)$ be a matching market, let P_W be responsive, and let $\succ_F$ be acyclical. Then, the worker-optimal stable mechanism, $\mu^W(P,q)$, is group strategy-proof and Pareto optimal.*

Proof. (a) First we prove group strategy-proofness. We prove the claim by contradiction. Let us assume that there exists a nonempty set of agents $V'\subset V$, P and $P'_{V'}=(P'_v)_{v\in V'}$ such that $\mu^W(P'_{V'},P_{V\setminus V'},q)(v)\,R_v\mu^W(P,q)(v)$ for every $v\in V'$ and $\mu^W(P'_{V'},P_{V\setminus V'},q)(v')\,P_{v'}\mu^W(P,q)(v')$ for some $v'\in V'$. Let $P''_{V'}=(P''_v)_{v\in V'}$ be such that, for every $v\in V'$, P''_v is responsive and coincides with P_v on the subsets of $\mu^W(P'_{V'},P_{V\setminus V'},q)(v)$ and ranks all subsets containing agents in $V\setminus\mu^W(P'_{V'},P_{V\setminus V'},q)(v)$ as unacceptable. We have $\mu^W(P''_{V'},P_{V\setminus V'},q)(v)=\mu^W(P'{}_{V'},P_{V\setminus V'},q)(v)$ for all $v\in V'$. The profile $P''_F=\left(P''_{V'\cap F},P_{F\setminus V'}\right)$ is responsive to the acyclical profile $\succ''_F$ of preferences over individual workers, where for every $f\in V'\cap F$, $\succ''_f$ coincides with $\succ_f$ on the workers belonging to $\mu^W(P''_{V'},P_{V\setminus V'},q)(f)$ and ranks all other workers as unacceptable and, for every $f\in F\setminus V'$, $\succ''_f=\succ_f$. Let $w_1,w_2,...,w_{|W|}$ be an order used to generate $\mu(P,q)=\mu^W(P,q)$ (see Proposition 2) as a serial dictatorship. Preferences $\succ_f$ and $\succ''_f$ coincide on the set of mutually acceptable workers and $A\left(f,\succ''_f\right)\subseteq A\left(f,\succ_f\right)$ for every $f\in F$. It follows that $\succ''_F$ is acyclical and $w_1,w_2,...,w_{|W|}$ can be used to generate an adjusted serial dictatorship leading to $\mu\left(P''_{V'},P_{V\setminus V'},q\right)=\mu^W\left(P''_{V'},P_{V\setminus V'},q\right)$. Let $w_s\in W$ be the maximal worker whose outcome is modified by the group deviation. Formally, let s be the minimal integer such that $\mu\left(P''_{V'},P_{V\setminus V'},q\right)(w_s)\neq\mu(P,q)(w_s)$. Since all workers with an index lower than s are matched to the same firms under $\mu\left(P''_{V'}, P_{V\setminus V'}, q\right)$ and under $\mu(P,q)$, we have $A_s\left(P''_{V'}, P_{V\setminus V'}, q\right) = A_s(P,q)$. It follows that $\mu\left(P''_{V'}, P_{V\setminus V'}, q\right)(w_s) = \mu(P,q)(w_s)$, which yields a contradiction.

(b) We prove the claim by contradiction. Assume that there exists a profile of preferences P and an individually rational matching ν such that $\nu(v)\, R_v \mu(P,q)(v)$ for every $v \in V$ and $\nu(v') P_{v'} \mu^W(P,q)(v') = \mu(P,q)(v')$ for some $v' \in V$. Let $v_s \in V'$ be the maximal worker who benefits from matching ν. Formally, let s be the minimal integer such that $\nu(w_s)\, P_{w_s} \mu(P,q)(w_s)$. All workers with an index lower than s are matched to the same firms under ν and under $\mu(P,q)$, and ν is individually rational, thus $\nu(w_s) \subseteq A_s(P,q)$. It follows that $\mu(P,q)(w_s) = C_{w_s}(A_s(P,q))\, R_s \nu(w_s)$, which yields a contradiction. ■

If we drop the assumption that workers' preferences are responsive, the mechanism $\mu(P,q)$ is still well defined and group strategy-proof for workers if the preferences of the firms are acyclical and responsive.

Proposition 3 *Let $M = (F, W, P, q)$ be a matching market, and let $\succ_F$ be acyclical. Then, the mechanism $\mu(P,q)$ is group strategy-proof for workers.*

The proof of Proposition 3 is very similar to the proof of Theorem 1 (part (a)) and thus omitted.

The reader can easily verify that any serial dictatorship is group strategy-proof for workers. However, the definition of the order $w_1, w_2, ..., w_{|W|}$ (thus acyclicity) and the restriction of worker w_s's choice to the subset A_s are crucial to guarantee the stability of the outcomes. First, if the order of moves is not defined properly, a serial dictatorship can produce allocations that are pairwise blocked. Furthermore, if the choice set of any of the workers is not restricted to the firms with vacant positions that she is acceptable to, a serial dictatorship can produce allocations that are not individually rational for firms.

The reader might wonder whether the incentive properties of the worker-optimal stable mechanism are a consequence of the existence of a unique stable matching. The following example shows that this is not the case. We present an example in which there exists a unique stable matching, there is a cycle in the preferences of the firms, and no stable mechanism is strategy-proof.

Example 1 *Let* $F = \{f_1, f_2, f_3\}$. *Let* $W = \{w_1, w_2, w_3\}$. *Set* $P_{w_1} : \{f_1, f_2\}, \{f_2\}, \{f_1\}$, $P_{w_2} : \{f_3\}, \{f_2\}$ *and* $P_{w_3} : \{f_1\}, \{f_3\}$. *Let* $w_1 \succ_{f_1} w_3 \succ_{f_3} w_2 \succ_{f_2} w_1$. *Let* $A(f_1) = \{w_1, w_3\}$, $A(f_2) = \{w_1, w_2\}$, $A(f_3) = \{w_2, w_3\}$. *Let* $q_f = 1$ *for all* $f \in F$. *There exists a unique stable matching* μ *where* $\mu(f_i) = \{w_i\}$ *for* $i = 1, 2, 3$. *If any stable mechanism is employed and worker* w_1 *reports preferences* $P'_{w_1} = \{f_2\}$, *she obtains a position at* f_2, *which she strictly prefers to* f_1. *It follows that truth telling is not a dominant strategy.*

Moreover, acyclicity is the minimal condition guaranteeing the existence of a stable and strategy-proof mechanism.

Proposition 4 *Assume that* $\succ_F$ *has a cycle. Then, there exist a profile of responsive preferences for workers* P_W *and a vector of quotas* q *for firms such that no stable mechanism is strategy-proof.*

Proof. Let $f_0, f_1, ..., f_T$, $w_0, w_1, ..., w_T$ such that $w_i \succ_{f_i} w_{i-1}$ for $i = 0, ..., T$ where $f_{-1} = f_T$. Set $q_f = 1$ for all $f \in F$. Set $P_{w_0} : \{f_1, f_0\}, \{f_1\}, \{f_0\}$, $P_{w_1} : \{f_2\}, \{f_1\}$, and set $P_{w_i} : \{f_{i+1}\}, \{f_i\}$ for $i = 1, 2, ..., T-1$. For all $w \notin \{w_0, w_1, ..., w_T\}$, let $\succ_w$ such that $A(w) \subseteq F \setminus \{f_0, f_1, ..., f_T\}$. Let φ be a stable mechanism. We have $\varphi(P, q)(f_i) = \{w_i\}$ for $i = 0, 1, ..., T$. Let $P'_{w_0} = \{f_1\}$. Then, $\{f_1\} = \varphi\left(P'_{w_0}, P_{-w_0}, q\right)(w_0)\, P_{w_0} \{f_0\} = \varphi(P)(w_0)$,which implies the claim. ■

Proposition 4 implies that, in a many-to-one setting, acyclicity of workers' preferences is the minimal condition guaranteeing that the firms' optimal stable mechanism is strategy-proof for firms.

4 Conclusions

In this paper, we prove that the worker-optimal stable mechanism is group strategy-proof and Pareto optimal in many-to-many matching markets when firms and workers have responsive preferences and the underlying preferences of the firms over individual workers are acyclical. If the preferences of the firms are acyclical but the preferences of the workers are not responsive we can use an Adjusted Serial Dictatorship. In this case our allocation will be group strategy-proof and efficient for workers.

Our result is especially relevant for environments in which firms' preferences can be interpreted as priorities. The restriction to acyclical priorities is particularly appropriate in the cases in which priorities reflect an underlying merit-based ranking. In this case, a serial dictatorship is an appealing implementation mechanism. It is a simple procedure, and can be considered fair if priorities are chosen in a "fair way" (see Ehlers and Klaus, 2003). This is often the case in multi-unit assignment problems such as the course allocation problem. In this case, acyclical priorities guarantee stability, efficiency and strategy-proofness without restrictions on the preferences of the workers.

The robustness of the results that we present and the flexibility that acyclical preferences provide make this property a relevant tool for market design.

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